University of California, Riverside
Department of Mathematics

# Final Exam

Mathematics 8B - First Year of Calculus
Sample 4

Instructions: This exam has a total of 200 points. You have 3 hours. You must show all your work to receive full credit You may use any result done in class. The points attached to each problem are indicated beside the problem. You are not allowed books, notes, or calculators. Answers should be written as $\sqrt{2}$ as opposed to 1.4142135....

The following formulae may be useful: for any $A, B$, we have

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B),\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B),\\
\tan(A+B) &= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}, \qquad \tan\left(\frac{A}{2}\right)=\frac{\sin(A)}{1+\cos(A)}=\frac{1-\cos(A)}{\sin(A)},\\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1-\cos(A)}{2}}, \qquad \cos\left(\frac{A}{2}\right)=\pm\sqrt{\frac{1+\cos(A)}{2}},\\
\sin(2A) &= 2\sin(A)\cos(A), \qquad \cos(2A)=\cos^2(A)-\sin^2(A),\\
\cos(2A) &= 2\cos^2(A)-1, \qquad \cos(2A)=1-2\sin^2(A),\\
\tan(2A) &= \frac{2A}{1-\tan^2(A)}
\end{aligned}$$

1. Find the precise values of

   (a) (5 points) $\sin\frac{4\pi}{3}$,

   (b) (5 points) $\tan\frac{\pi}{12}$,

   (c) (5 points) $\cot 15^\circ$,

   (d) (5 points) $\arccos\frac{\sqrt{3}}{2}$.

2. Let

$$g(x) = \begin{cases} \frac{\cos x - 1}{x} & x \neq 0 \\ a & x = 0 \end{cases}$$

(a) (10 points) Using known theorems, show that the function $g(x)$ is continuous at all points $x \neq 0$.

(b) (10 points) For which value of $a$ is the function also continuous at $x = 0$?

3. Evaluate the following limits:

(a) (10 points) $\lim_{x \to 0} \frac{\sin x}{x^3}$

(b) (10 points) $\lim_{x \to \infty} \left(\sqrt{x+1} - \sqrt{x}\right)$.

4. Find the rules for differentiation to find derivatives of the following functions:

(a) (10 points) $f(x) = \sin^3(x) + \sin(x^3) + \sin(3x)$

(b) (10 points) $\frac{\sin x}{x^3}$.

5. (20 points) Let $f(x)$ be a differentiable function that satisfies the equations

$$f(x) + (f(x) + 1)^4 = 1 + x$$

and

$$f(0) = 0$$

Find the equation of the tangent line to $f(x)$ at $x = 0$.

6. Let $f(x) = 2x^3 - 9x^2 + 12x$.

(a) (4 points) Calculate the first derivative $f'(x)$ and find all values of $x$ so that $f'(x) = 0$.

(b) (4 points) Calculate the second derivative $f''(x)$ and find all values of $x$ so that $f''(x) = 0$

(c) (4 points) Use $f'(x)$ to locate critical points of $f(x)$ and determine where $f(x)$ is monotone increasing and monotone decreasing.

(d) (4 points) Use $f''(x)$ to determerme where $f(x)$ is concave up and concave down and to identify its points of inflection.

(e) (4 points) Sketch the graph of $f(x)$. The graph should exhibit all local extrema and points of inflection, and reflect monotonicity and concavity correctly.

7. Find the most general antiderivatives of the following functions:

   (a) (10 points) $f(x) = x^3 + \sin x - 3$

   (b) (10 points) $f(x) = (1 + 3x)^4$

8. Let $f(x) = \sqrt{x - 3}$

   (a) (10 points) Use the definition of the derivative to calculate $f'(x)$

   (b) (10 points) Find the equation of the tangent line to the graph of $f(x)$ at $x = 4$.

9. Let $p(x) = x^3 - 3x - 3$.

   (a) (10 points) Use to Intermediate Value Theorem to prove that $p(x)$ has a root in the interval.

   (b) (10 points) Use Rolle's Theorem to prove that $p(x)$ has exactly one root in the interval $[2, 3]$.

10. A spherical balloon is inflated with helium at the rate of $144\pi$ cubic feet per minute.

   (a) (10 points) How fast is the balloon'r radius increasing at the instant the radius is 6 feet?

   (b) (10 points) How fast is the surface area increasing at the instant the radius is 6 feet?